Introduction to Euclid's Geometry

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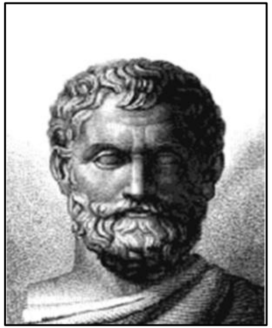
Introduction

- The word ‘geometry’ comes form the Greek words ‘geo’, meaning the ‘earth’, and ‘metrein’, meaning ‘to measure’. Geometry seems to have originated from the need for measuring land. This branch of mathematics was studied in various forms in every ancient civilisation, be it in Egypt, Babylonia, China, India, Greece, etc. The people of these civilisations faced many practical problems which required the development of geometry in various ways.

- Greek mathematician, Thales is credited with giving the first known proof. This proof was of the statement that a circle is bisected (i.e., cut into two equal parts) by its diameter. One of Thales’ most famous pupils was Pythagoras (572 BCE), whom you have heard about. Pythagoras and his group discovered many geometric properties and developed the theory of geometry to a great extent. This process continued till 300 BCE. At that time Euclid, a teacher of mathematics at Alexandria in Egypt, collected all the known work and arranged it in his famous treatise, called ‘Elements’. He divided the ‘Elements’ into thirteen chapters, each called a book. These books influenced the whole world’s understanding of geometry for generations to come.



Thales (640 BCE-546 BCE)

Euclid (325 BCE-265 BCE)

Euclid’s Definitions, Axioms and Postulates

- Starting with his definitions, Euclid assumed certain properties, which were not to be proved. These assumptions are actually ‘obvious universal truths’. He divided them into two types: axioms and postulates. He used the term ‘**postulate’** for the assumptions that were specific to geometry. Common notions (often called **axioms**), on the other hand, were assumptions used throughout mathematics and not specifically linked to geometry.

- Though Euclid defined a point, a line, and a plane, the definitions are not accepted by mathematicians. Therefore, these terms are now taken as undefined.

- **Theorems** are statements which are proved, using definitions, axioms, previously proved statements and deductive reasoning.

Euclid’s Definitions, Axioms and Postulates (Contd..)

- Some of **Euclid’s axioms** were :

  (1) Things which are equal to the same thing are equal to one another.

  (2) If equals are added to equals, the wholes are equal.

  (3) If equals are subtracted from equals, the remainders are equal.

  (4) Things which coincide with one another are equal to one another.

  (5) The whole is greater than the part.

  (6) Things which are double of the same things are equal to one another.

  (7) Things which are halves of the same things are equal to one another.

Euclid's Definitions, Axioms and Postulates (Contd..)

- Let's discuss Euclid's postulates:

  Postulate 1 : A straight line may be drawn from any one point to any other point.

  Note that this postulate tells us that at least one straight line passes through two distinct points, but it does not say that there cannot be more than one such line. However, in his work, Euclid has frequently assumed, without mentioning, that there is a *unique* line joining two distinct points.

  Postulate 2 : A terminated line can be produced indefinitely.

  Note that what we call a line segment now-a-days is what Euclid called a terminated line. So, according to the present day terms, the second postulate says that a line segment can be extended on either side to form a line (see Fig. 1.1)

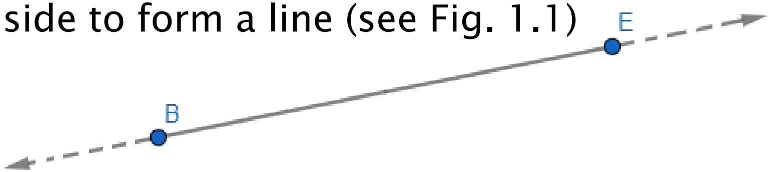


Fig 1.1

Euclid's Definitions, Axioms and Postulates (Contd..)

Postulate 3 : A circle can be drawn with any centre and any radius.

Postulate 4 : All right angles are equal to one another.

Postulate 5 : If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles. (See Fig. 1.2)

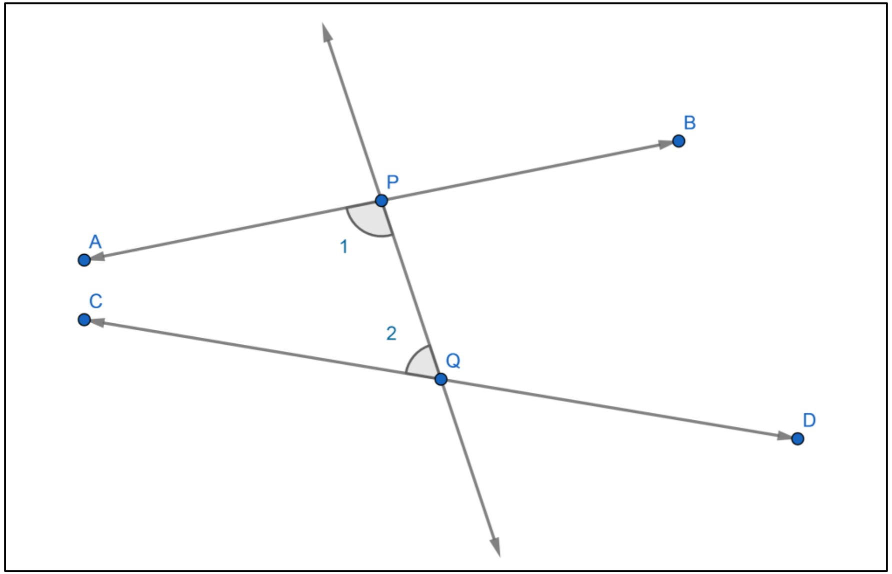


Fig. 1.2

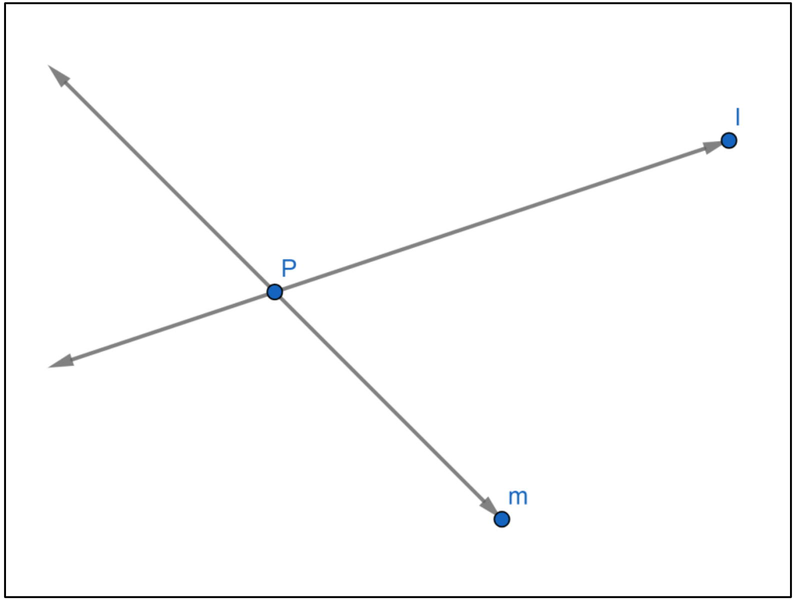
Euclid's Definitions, Axioms and Postulates (Contd..)

<u>Theorem 1</u>: Two distinct lines cannot have more than one point in common.

<u>Proof</u>: Here we are given two lines *l* and *m.* We need to prove that they have only one point in common. (See Fig. 1.3)

For the time being, let us suppose that the two lines intersect in two distinct points, say P and Q. So, you have two lines passing through two distinct points P and Q. But this assumption clashes with the axiom that only one line can pass through two distinct points. So, the assumption that we started with, that two lines can pass through two distinct points is wrong.

So, we conclude that two distinct lines cannot have more than one point in common.



<u>Fig. 1.3</u>

Equivalent Versions of Euclid's Fifth Postulate

- **Euclid's fifth postulate** states that: "*If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.*"

- There are several equivalent versions of this postulate. One of them is **'Playfair's Axiom'** (given by a Scottish mathematician John Playfair in 1729), as stated below:

  '*For every line l and for every point P not lying on l, there exists a unique line m passing through P and parallel to l*'.

  From Fig. 1.4, you can see that of all the lines passing through the point P, only line m is parallel to line l.

  This result can also be stated in the following form:
  'Two distinct intersecting lines cannot be parallel to the same line.'

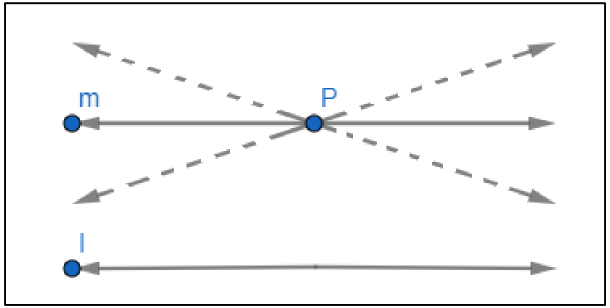


Fig. 1.4

Equivalent Versions of Euclid’s Fifth Postulate (Contd..)

- Euclid did not require his fifth postulate to prove his first 28 theorems. Many mathematicians, including him, were convinced that the fifth postulate is actually a theorem that can be proved using just the first four postulates and other axioms. However, all attempts to prove the fifth postulate as a theorem have failed. But these efforts have led to a great achievement – the creation of several other geometries. These geometries are quite different from Euclidean geometry. They are called *non-Euclidean geometries.*

Summary

- Though Euclid defined a point, a line, and a plane, the definitions are not accepted by mathematicians. Therefore, these terms are now taken as undefined.
- **Axioms or postulates** are the assumptions which are obvious universal truths. They are not proved.
- **Theorems** are statements which are proved, using definitions, axioms, previously proved statements and deductive reasoning.
- **Some of Euclid's axioms were** :

(1) Things which are equal to the same thing are equal to one another.

(2) If equals are added to equals, the wholes are equal.

(3) If equals are subtracted from equals, the remainders are equal.

(4) Things which coincide with one another are equal to one another.

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(7) Things which are halves of the same things are equal to one another.

Summary (Contd..)

- Euclid's postulates were :

  Postulate 1 : A straight line may be drawn from any one point to any other point.

  Postulate 2 : A terminated line can be produced indefinitely.

  Postulate 3 : A circle can be drawn with any centre and any radius.

  Postulate 4 : All right angles are equal to one another.

  Postulate 5 : If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.

- Two equivalent versions of Euclid's fifth postulate are:

  (i) 'For every line *l and for every point P not lying on l, there exists a unique line m* passing through P and parallel to *l'.*

  (ii) Two distinct intersecting lines cannot be parallel to the same line.

- All the attempts to prove Euclid's fifth postulate using the first 4 postulates failed. But they led to the discovery of several other geometries, called non-Euclidean geometries.

THANK YOU